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A Comparison of Primal and Dual Methods
of Linear Programming

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Victor Klee

A COMPARISON OF PRIMAL AND DUAL METHODS

OF LINEAR PROGRAMMING

by

Victor Klee

ABSTRACT

Both primal and dual methods of linear programming consist of
relatively efficient ways of searching among certain sets of points
for one at which an extreme value of a linear form is attained. For
a given problem the primal search is applied to one set of points
and the dual search to another set. It is of interest to compare the
possible cardinalities of these sets for guidance as to which method
is preferable in various circumstances. Let $\pi(d,n)$ and $\Pi(d,n)$
denote respectively the minimum and the maximum of the cardinality
of the set to which the primal search is applied, when the feasible
region is a d-dimensional polyhedron having n facets and reasonable
nondegeneracy conditions are satisfied. Let $\delta(d,n)$ and $\Delta(d,n)$ be
similarly defined for dual searches. Various results on these num-
bers are obtained, leading in particular to the conjecture that
$\Pi(d,n) \leq \Delta(d,n)$ when $n \geq 2d$, while $\Pi(d,n) \geq \Delta(d,n)$ when $n \leq 2d$.

Setting of the problem

We are concerned with procedures for minimizing a linear form ϕ_0 over a convex polyhedron P defined by a finite system of linear inequalities —— say

$$P = \{x \epsilon R^d : \phi_1(x) \leq \alpha_1, \phi_2(x) \leq \alpha_2, \cdots, \phi_n(x) \leq \alpha_n\},$$

where the ϕ_i's are linear forms on the d-dimensional real vector space R^d. We assume the following nondegeneracy conditions, some of which are inessential but make for a simpler discussion:

(1) P is of dimension d;

(2) the system of inequalities $\phi_i(x) \leq \alpha_i$ $(1 \leq i \leq n)$ is minimal with respect to determining P;

(3) the linear form ϕ_0 attains a minimum on P;

(4) $n \geq d$; each set of d functions ϕ_i $(0 \leq i \leq n)$ is a linear basis for R^d (regarded as self-dual in the usual way);

(5) no point lies on more than d of the hyperplanes $H_i = \{x : \phi_i(x) = \alpha_i\}$, $1 \leq i \leq n$.

We do not assume P is bounded.

Under the conditions (1)-(5) it is true that each d of the hyperplanes H_i have a unique common point, the polyhedron P has exactly d edges incident to each vertex, and the minimum of ϕ_0 on P is attained at a unique point of R^d which satisfies the following two conditions:

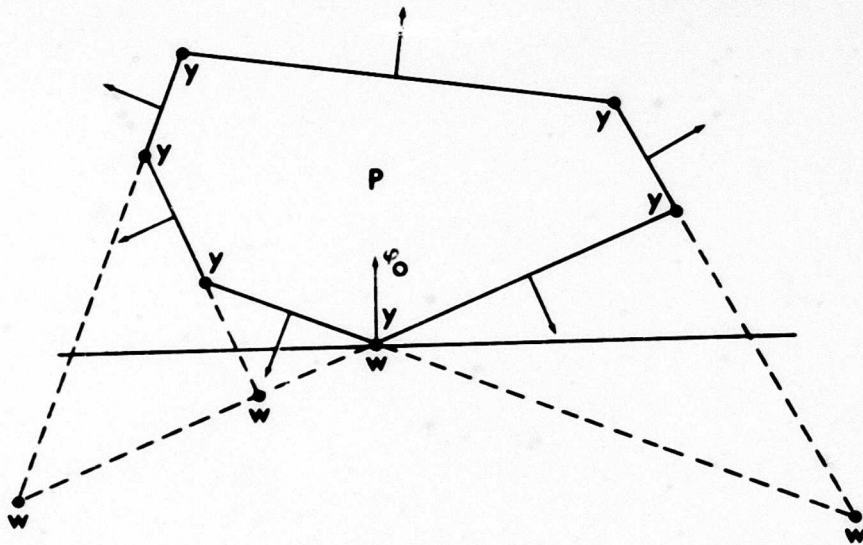(6) it is the intersection of d hyperplanes $H_{k(1)}, \cdots, H_{k(d)}$ which determine a point of P;

(7) it is the intersection of d hyperplanes $H_{k(1)}, \cdots, H_{k(d)}$ such that the set $\{\phi_0, \phi_{k(1)}, \cdots, \phi_{k(d)}\}$ is a positive basis for R^d.

Condition (6) merely requires the point to be a vertex of P in the usual geometric sense or, as we may say here for emphasis, a *primal vertex*. A point satisfying condition (7) will be called a *dual vertex* for the problem of minimizing ϕ_0 on P. (In the figure below, the dual vertices are denoted by w and the primal vertices by y; the linear forms ϕ_i are represented by vectors orthogonal to their level sets.) By a *primal* [resp. *dual*] method of linear programming we mean any procedure which searches for the minimizing point among the primal [resp. dual] vertices. (Primal methods are discussed in all treatments of linear programming. Some of the better treatments of dual methods are those of Cheney & Goldstein [2], Dantzig [3], Hadley [11], Lemke [14] and Wagner [16].) For each dual vertex w and primal vertex y,

$$\phi_0(w) \le \min \phi_0 P \le \phi_0(y).$$

Thus the dual methods approach the minimizing value and vertex from below, the primal methods from above.

Under our assumptions there are exactly $\binom{n}{d}$ points of R^d which are determined by the various sets of d hyperplanes H_i, but of course the number of primal vertices and the number of dual vertices will vary with P and ϕ_0. It is of interest to estimate these numbers and to compare them, for their relative magnitudes are helpful in deciding whether

Primal and dual vertices for a linear program.

a primal or a dual method is preferable for a given problem. Let $\pi(d,n)$ and $\Pi(d,n)$ denote respectively the minimum and the maximum of the number of primal vertices, and let $\delta(d,n)$ and $\Delta(d,n)$ denote respectively the minimum and the maximum of the number of dual vertices, these extrema being taken over all $(n+1)$-tuples $(\phi_0, \phi_1, \cdots, \phi_n)$ of linear forms on R^d which satisfy the nondegeneracy conditions (1)-(5). By applying known methods and results concerning convex polyhedra we are able to prove the following theorems.

THEOREM. $\pi(d,n) = n-d+1; \qquad \delta(d,n) = 1.$

THEOREM.

$$(8) \qquad \Pi(d,n) \geq \binom{n - \left\lceil \frac{d+1}{2} \right\rceil}{n-d} + \binom{n - \left\lceil \frac{d+2}{2} \right\rceil}{n-d},$$

with equality if $d \leq 8$ or $n \leq d+3$ or $n \geq (d/2)^2 - 1.$

$$(9) \qquad \Delta(d,n) \geq \binom{n - \left\lceil \frac{n-d+1}{2} \right\rceil}{d} + \binom{n - \left\lceil \frac{n-d+2}{2} \right\rceil}{d},$$

with equality if $d \leq 3$ or $n \leq d+8$ or $n \leq d+2+2(2d+1)^{\frac{1}{2}}.$

It is conjectured that equality always holds in (8) and (9).

Corollary.

$\Pi(d,n) \leq \Delta(d,n)$ if $n \geq 2d$ and equality holds in (8).

$\Pi(d,n) \geq \Delta(d,n)$ if $n \leq 2d$ and equality holds in (9).

For a variety of reasons, these results would not lead to a clear choice between primal and dual methods in practical problems, even if

equality were known in both (8) and (9). However, the cardinality of the set in which a search is to be carried out is surely relevant to the difficulty of the search, and since the common procedures for primal and dual methods are very similar in structure it seems the above comparisons should be useful.

I am indebted to A.A. Goldstein and M. Perles for helpful comments.

The number of vertices of a simple polyhedron

Henceforth we assume d and n are integers with $n > d \geq 1$. By *polyhedron* we mean a subset of R^d which is the intersection of a finite number of closed halfspaces; a *polytope* is a bounded polyhedron. Dimensions are indicated by prefixes, and the 0-faces, 1-faces and $(d-1)$-faces of a d-polyhedron are called respectively its *vertices*, *edges* and *facets*. A polyhedron is said to be *pointed* provided it has at least one vertex, and a d-polyhedron is *simple* if it is pointed and each of its vertices is on exactly d edges or, equivalently, on exactly d facets. A *polyhedron of class* (d,n) is a pointed d-polyhedron with exactly n facets.

The nondegeneracy conditions of the preceding section imply P is a simple polyhedron of class (d,n). Conversely, each simple polyhedron of class (d,n) can be deformed (by a slight displacement of its facets) into a combinatorially equivalent polyhedron which satisfies the nondegeneracy conditions. Hence $\pi(d,n)$ and $\Pi(d,n)$ are respectively the minimum and the maximum number of vertices of simple polyhedra of

class (d,n). The stated lower bound for π was established by Gale [8], who showed [9] that equality holds when $n \leq d+3$. Equality when $n \geq (d/2)^2-1$ was proved by Klee [12]. These results establish equality when $d \leq 6$, extended to $d \leq 8$ by Grünbaum [10] using theorems of Fieldhouse [6] and Kruskal [13]. This takes care of (8), and we want next to show that $\pi(d,n) = n-d+1$. The corresponding problem for *polytopes* of class (d,n) is still open, although it may eventually be settled by a proof of the following

Conjecture. *For simple polytopes of class* (d,n) *the minimum number of vertices is* (d-1)n - (d-2)(d+1).

For $d = 4$ this was stated by Brückner [1] as a theorem, but Steinitz [15] observed Brückner's proof was incorrect. The conjecture is obviously correct (by Euler's theorem) for $d \leq 3$ and has been proved by Grünbaum [10] for $n \leq d+3$. Although the general case is unsettled, we find it convenient to define a *minimal* (d,n)-*polytope* as a simple polytope of class (d,n) which has (d-1)n - (d-2)(d+1) vertices.

Lemma. *For* 1<d<n *there is a sequence of polytopes* $P_2 \subseteq P_3 \subseteq \cdots \subseteq P_d$ *such that each* P_j *is a minimal* (j, n-d+j)-*polytope and is a face of* P_d.

Proof. For $n = d+1$ let P_d be a d-simplex and let P_j be a facet of P_{j+1} for $2 \leq j < d$. Then for fixed d proceed by induction on n. Having constructed the desired sequence $P_2' \subseteq P_3' \subseteq \cdots \subseteq P_d'$ for $n-1$, choose a vertex x of P_2' and a closed halfspace Q in the affine hull of P_d' such that Q does not include x but all other vertices of P_d' are interior to Q. Let $P_j = P_j' \cap Q$ for $2 \leq j \leq d$.

A function f will be called a d-*fold adjacency function* for a
graph G provided the following three conditions are satisfied:

(10) the domain of f is the set V of all vertices of G;

(11) for each vertex v of G, f(v) is a set of cardinality d;

(12) whenever v and w are adjacent vertices of G there are
d-1 points common to the sets f(v) and f(w).

Lemma. *If* V *is the set of all vertices of a connected graph* G
and f *is a* d-*fold adjacency function for* G *then*

$$\text{card} \cup_{v \in V} f(v) \le \text{card } V + d - 1.$$

Proof. Use induction on card V, noting that the points of V
can be linearly ordered so that each is adjacent to at least one of its
predecessors. (This argument, due to M. Perles, is simpler than my
original proof.)

It would be of interest to sharpen the above inequality for various
restricted classes of connected graphs, in particular for the d-connected
d-valent graphs.

THEOREM. *For simple polyhedra of class* (d,n) *the minimum number
of vertices is* n-d+1.

Proof. Let us first construct a simple d-polyhedron P which
has n facets, all unbounded, and has n-d+1 vertices. For this purpose
we appeal to the first lemma to obtain a minimal (d,n+1)-polytope P_d
having a minimal (d-1,n)-polytope P_{d-1} as one of its facets. We may

assume P_d is in R^d. Let P be the unbounded polyhedron obtained from P_d by "removing" P_{d-1}; that is, P is the image of P_d under a projective transformation which carries P_{d-1} into the hyperplane at infinity. Then P is a simple polyhedron of class (d,n) and since (with the P_i's as constructed in the lemma) every facet of P_d intersects P_{d-1} it follows that every facet of P is unbounded. The vertices of P correspond to vertices of P_d which are not in P_{d-1}, and the number of such vertices is

$$[(d-1)(n+1) - (d-2)(d+1)] - [(d-2)n - (d-3)d] = n-d+1.$$

To complete the proof, consider an arbitrary simple polyhedron P of class (d,n). Let V denote the set of all vertices of P, and G the graph formed by the vertices and bounded edges of P. That G is connected is well-known and follows, for example, from a basic theorem of linear programming. For each $v \epsilon V$ let $f(v)$ denote the set of all facets of P incident to v. Since P is a *simple* d-polyhedron, f is a d-fold adjacency function for G, and since each facet is incident to at least one vertex it follows from the second lemma that

$$n \leq \text{card } V + d - 1.$$

The number of positive bases contained in a Haar set

A *positive basis* for R^d is a subset B of R^d which positively spans R^d and is positively independent; that is, each point of R^d is a positive combination of points from B but no point p of B is a

positive combination of points from $B \sim \{p\}$. A subset of R^d will be
called a *Haar set* provided it includes at least d points and each of
its d-pointed subsets is a linear basis for R^d. Any positive basis con-
tained in a Haar set is of cardinality $d+1$ and in fact is the set of
all vertices of a d-simplex whose interior includes the origin 0; con-
versely, each such set of vertices is both a positive basis and a Haar
set.

In addition to the dual methods of linear programming there are
algorithms for convex programming which deal with positive bases con-
tained in Haar sets (see Cheney & Goldstein [2] and some of their ref-
erences; also Descloux [5]). For each subset X of R^d let $b(X)$ denote
the number of positive bases for R^d contained in X. Let $m_H(d,n)$
and $M_H(d,n)$ denote respectively the minimum and the maximum of $b(X)$
as X ranges over all Haar sets of cardinality n which positively
span R^d.

THEOREM.

$$(13) \qquad m_H(d,n) = n - d.$$

$$(14) \qquad M_H(d,n) \geq \binom{n - \left\lceil \frac{n-d}{2} \right\rceil}{d+1} + \binom{n - \left\lceil \frac{n-d+1}{2} \right\rceil}{d+1},$$

with equality if $d \leq 2$ *or* $n \leq d+9$ *or* $n \leq d+3+2(2d+3)^{\frac{1}{2}}$.

It is conjectured that equality always holds in (14). The equality (13)
was proved by Cheney & Goldstein (§§13-14 of [2]) but is included here
for the sake of completeness.

Proof. Let Y be a Haar set consisting of the d points of a linear basis for R^d together with $n-d$ strictly negative combinations of these points. Then $m_H(d,n) \leq b(Y) = n-d$.

Now consider an arbitrary Haar set X consisting of n points $x_1, \cdots, x_n$ which positively span R^d, and let C denote the set of all convex relations on X. That is, C is the set of all n-tuples $(\gamma_1, \cdots, \gamma_n)$ of real numbers satisfying the following conditions:

(15) $\quad \gamma_i \geq 0, \; i = 1, \cdots, n,$

(16) $\quad \sum_1^n \gamma_i = 1,$

(17) $\quad \sum_1^n \gamma_i x_i = 0,$

where of course the 0 in (17) is the origin of R^d. For $k = 1,2,3$ let C_i denote the set of all points $(\gamma_1, \cdots, \gamma_n)$ of R^n which satisfy condition $(14 + i)$. Then C_1 is the positive orthant of R^n, C_2 is a hyperplane in $R^n \sim \{0\}$, and since R^d is linearly spanned by X the set C_3 is a linear subspace of deficiency d in R^n. By a theorem of Davis (3.6 of [4]) the set X admits a strictly positive relation and hence C includes an interior point of C_1. Since $C = C_1 \cap C_2 \cap C_3$ we conclude C is an $(n-d-1)$-polytope whose facets are all of the form $F_i = \{\gamma \epsilon C : \gamma_i = 0\}$. Recalling that X is a Haar set, we see that C is simple, and a set F_i is a facet of C if and only if F_i is nonempty. From this or from another theorem of Davis (4.4 of [4], which applies to arbitrary positively spanning sets) it follows that the vertices of C are exactly the convex relations supported by the various positive bases contained in X.

Since C is a polytope of dimension $n-d-1$ it has at least $n-d$ vertices and thus X contains at least $n-d$ positive bases. Hence $m_H(d,n) = n-d$. And since C has at most n facets it follows that $M_H(d,n) \leq \Pi(n-d-1,n)$.

With

$$G(d,n) = \binom{n - \left\lceil \frac{n-d}{2} \right\rceil}{d+1} + \binom{n - \left\lceil \frac{n-d+1}{2} \right\rceil}{d+1},$$

a result stated earlier implies $\Pi(n-d-1,n) = G(d,n)$ if

$$n-d-1 \leq 8 \quad \text{or} \quad n \leq (n-d-1) + 3 \quad \text{or} \quad n \geq \left(\frac{n-d-1}{2}\right)^2 - 1,$$

and these conditions are equivalent respectively to

$$n \leq d+9, \quad d \leq 2, \quad \text{and} \quad n \leq d+3+2(2d+3)^{\frac{1}{2}}.$$

To complete the proof it remains to show $M_H(d,n) \geq G(d,n)$ ———— that is, to construct a Haar set of cardinality n in R^d which contains $G(d,n)$ positive bases. This may be done by an inductive procedure similar to one used by Gale (Theorem III of [7]) for a closely related purpose. Alternatively, from the existence of simple polytopes of class $(n-d-1,n)$ which have $G(d,n)$ vertices (Gale [8]) it is possible to arrive at the desired Haar set by means of a correspondence employed by Gale (p. 259 of [7]) and studied in more detail by M. Perles. (See the disucssion of Gale-diagrams in [10]).

The number of dual vertices

In §§17-19 of [2] Cheney & Goldstein discuss an algorithm for minimizing a polyhedral convex function ϕ over R^d, the function being assumed to have the form

$$(18) \qquad \psi(x) = \max_{0 \le i \le k} (\phi_i(x) - \alpha_i)$$

where the α_i's are constants and the set $\{\phi_o, \cdots, \phi_k\}$ is a Haar set of linear forms positively spanning R^d. For this unconstrained minimum problem their algorithm calls for a search in the set of positive bases for R^d contained in $\{\phi_o, \cdots, \phi_k\}$. Results of the preceding section are of interest as estimates of the cardinality of this set. In §§23-24 of [2] Cheney & Goldstein describe an algorithm for minimizing ψ (as in (18)) over a polyhedron $\{x \epsilon R^d : \phi(x) \le 0\}$, where

$$\phi(x) = \max_{k \le i \le n} \phi_i(x) - \alpha_i$$

and the set $\{\phi_o, \cdots, \phi_k, \phi_{k+1}, \cdots, \phi_n\}$ is assumed to be a Haar set positively spanning R^d. Here the search is confined to the positive bases which are contained in $\{\phi_o, \cdots, \phi_n\}$ and include at least one of the functions $\phi_o, \cdots, \phi_k$. This suggests a study of numbers $b(X,Y)$ where Y is a subset of X and $b(X,Y)$ is the number of positive bases which intersect Y and are contained in X. For $1 \le d < n$ and $1 \le k \le n$ let $m_H^k(d,n)$ and $M_H^k(d,n)$ denote respectively the minimum and the maximum of $b(X,Y)$ as X ranges over all Haar sets of cardinality n which positively span R^d and Y is a k-pointed subset of X. Of course $m_H^k(d,n) = m_H(d,n)$ and $M_H^k(d,n) = M_H(d,n)$ for $k \ge n - d$. It can be verified also that $m_H^1(d,n+1) = \delta(d,n)$ and $M_H^1(d,n+1) = \Delta(d,n)$.

THEOREM. $\quad m_H^k(d,n) = \begin{cases} n-d & \underline{for} \quad n-d \le k \le n \\ k & \underline{for} \quad 1 \le k \le n-d \end{cases}$

Proof. To see the claimed maximum is achieved when $1 \le k < n-d$, let X be a Haar set consisting of the d points of a linear basis for R^d together with $n-d$ strictly negative combinations of these points, and let Y consist of k of these combinations. Then $b(X,Y) = k$. To see $m_H^k(d,n) \ge k$ when $1 \le k < n-d$, consider an arbitrary Haar set $X = \{x_1, \cdots, x_n\}$ in R^d and its subset $Y = \{x_1, \cdots, x_k\}$. Let the $(n-d-1)$-polytope C be defined as in the preceding section and let

$$F = \{\gamma \varepsilon C : \gamma_1 = \gamma_2 = \cdots = \gamma_k = 0\}.$$

Then F is a face of dimension $\le n-d-k-1$ and hence C can be mapped onto a k-polytope K by an affine transformation which carries F into a single vertex of K. There are at least k additional vertices of K and each of them is the image of at least one vertex of C. Thus C has at least k vertices $\gamma = (\gamma_1, \cdots, \gamma_n)$ such that $\gamma_i \ne 0$ for some i with $1 \le i \le k$, and each such vertex corresponds to a positive basis for R^d which intersects Y and is contained in X.

A special and obvious case of the theorem just proved is that $\delta(d,n) = 1$.

For $1 < k < n-d$ our results on the numbers $M_H^k(d,n)$ are not very satisfactory. However, the following result shows that $\Delta(d,n) = M_H(d-1,n)$, and in conjunction with the preceding section this justifies the estimate given earlier for $\Delta(d,n)$.

THEOREM. $M^1_H(d,n) = M_H(d-1,n-1)$.

Proof. To see that $M^1_H(d,n) \geq M_H(d-1,n-1)$, let W be a Haar set of cardinality $n-1$ in R^{d-1} containing $M_H(d-1,n-1)$ positive bases. With R^{d-1} embedded in R^d as a hyperplane through the origin, choose $u_d \epsilon R^d \sim R^{d-1}$ and let

$$X = \{w-u_d : w \epsilon W\} \cup \{u_d\}.$$

Then X is a Haar set and $b(X, \{u_d\}) = b(W)$.

To see that $M^1_H(d,n) \leq M_H(d-1,n-1)$, consider an arbitrary Haar set $\{x_1, \cdots, x_n\}$ of cardinality n in R^d. Define C as previously and let $F_n = \{\gamma \epsilon C : \gamma_n = 0\}$. If F_n is empty then x_n is not in the positive span of $\{x_1, \cdots, x_{n-1}\}$ and there is a hyperplane R^{d-1} through 0 which strictly separates x_n from $\{x_1, \cdots, x_{n-1}\}$. If t is a linear projection of R^d onto R^{d-1} which carries x_n onto the origin, then $\{tx_1, \cdots, tx_{n-1}\}$ is a Haar set in R^{d-1} with

$$b(\{x_1, \cdots, x_n\}, \{x_n\}) = b(\{tx_1, \cdots, tx_{n-1}\}) \leq M_n(d-1,n-1).$$

In the remaining case F_n is a facet of C. The number $b(\{x_1, \cdots, x_n\}, \{x_n\})$ is equal to the number of vertices of C which are not in F_n and hence (considering a projective transformation which sends F_n into the hyperplane at infinity) is equal to the number of vertices of a simple $(n-d-1)$-polyhedron with at most $n-1$ facets. The maximum of such numbers is $\Pi(n-d-1,n-1) = M_H(d-1,n-1)$.

Now that the stated estimates for $\Pi(d,n)$ and $\Delta(d,n)$ have been established, only the comparison of the two numbers remains. Note first

that the conjectured values for $\Pi(d,2d)$ and $\Delta(d,2d)$ are equal. For $n \neq 2d$ the details of the comparison depend on the parity of d and n, but as the idea is the same for all four cases we discuss only the case in which d is odd and n is even —— say $d = 2j-1$ and $n-d = 2k-1$, so that $n-2d = 2(k-j)$. We want to compare the numbers

$$P(j,k) = \binom{j+2k-2}{2k-1} \quad \text{and} \quad D(j,k) = \binom{k+2j-2}{2j-1},$$

which are half the conjectured values for $\Pi(d,n)$ and $\Delta(d,n)$ respectively. Let

$$R(j,k) = \frac{P(j,k)}{D(j,k)} = \frac{(j+2k-2)!\,(2j-1)!\,(k-1)!}{(k+2j-2)!\,(2k-1)!\,(j-1)!}$$

Then $R(j,j) = 1$,

$$\frac{R(j,k)}{R(j,k+1)} = \frac{2(2k+1)(k+2j-1)}{(j+2k)(j+2k-1)} \quad \text{and} \quad \frac{R(j,k)}{R(j+1,k)} = \frac{(k+2j)(k+2j-1)}{2(2j+1)(j+2k-1)} .$$

Thus

$$\frac{R(j,k)}{R(j,k+1)} > 1 \quad \text{for} \quad k > \frac{j}{4} - \frac{5}{4} + \frac{1}{2j} , \quad \frac{R(j,k)}{R(j+1,k)} < 1 \quad \text{for} \quad j > \frac{k}{4} - \frac{5}{4} + \frac{1}{2k} .$$

If $n>2d$ then $k>j$ and

$$\frac{1}{R(j,k)} = \frac{R(j,j)}{R(j,j+1)} \frac{R(j,j+1)}{R(j,j+2)} \cdots \frac{R(j,k-1)}{R(j,k)} > 1,$$

whence $P(j,k) < D(j,k)$. If $n < 2d$ then $j > k$ and

$$\frac{1}{R(j,k)} = \frac{R(k,k)}{R(k+1,k)} \frac{R(k+1,k)}{R(k+2,k)} \cdots \frac{R(j-1,k)}{R(j,k)} < 1,$$

whence $P(j,k) > D(j,k)$.

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